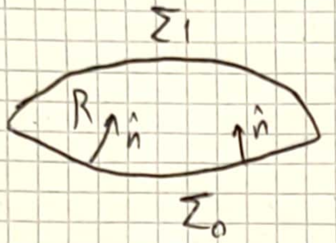
General Relativity $\boxed{\text{Week 11}}$

Last time: We saw the proof of the well-posedness of the IVP for the (inhomogeneous) covariant wave equation



$$\begin{cases} \Box_g \psi = F & \text{on } R \\ \psi|_{\Sigma_0} = \psi_0, \quad \hat{n}(\psi)|_{\Sigma_0} = \psi_1 \end{cases}$$

Existence: By reducing to the case where $\psi_0 = \psi_1 = 0$ and using Hahn-Banach.

An alternative proof of existence (see the book by Taylor):

Again assume w.l.o.g that $\psi_0 = \psi_1 = 0$.

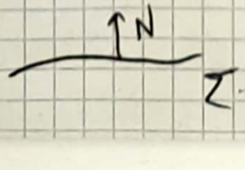
We're going to use the following existence result in the real analytic category:

Cauchy-Kovaleskaya theorem:

The initial value problem for a linear PDE with (real) analytic coefficients, analytic initial data on an analytic hypersurface which is $\boxed{\text{not characteristic}}$ has an analytic solution. Moreover, the radius of analyticity of the solution can be bounded from below by a constant depending on the radii of analyticity of the data, the coefficients etc.

For the wave equation: Non-characteristic surfaces: spacelike/timelike

For elliptic eqn: Every hypersurface is not characteristic

(In general:  is non-characteristic for 2^{nd} order PDE if $N^2(\psi)$ is computable in terms of ψ_0, ψ_1)

Note: Cauchy - Kovalevskaya is "deceivingly" general:

For instance, for $\Delta\psi = 0$: The initial value problem with data on $\{x=0\}$ has no solution if the data are not real-analytic.

Going back to the IVP
$$\begin{cases} \Box_g \psi = F \\ \psi|_{\Sigma_0} = 0, \quad \hat{n}(\psi)|_{\Sigma_0} = 0 \end{cases} \quad (1)$$

~~Assume~~ Let us approximate (g, F) in C^k, $k \gg 1$ by a sequence of (g_n, F_n) which are real analytic with infinite-radius of analyticity. (e.g. $F_n = F * n^d e^{-\frac{|x|^2 n}{2}}$, $d = \dim M$).
convolution

~~Assume wlog that ...~~

Then Cauchy-Kovalevskaya provides a sequence of solutions ψ_n with uniform radius of analyticity — assume w.l.o.g. that all of the domain R is contained within the ball of analyticity.

of $\begin{cases} \Box_{g_n} \psi_n = F_n \\ \psi_n|_{\Sigma_0} = 0, \dots \end{cases}$

We will show that $\psi_n \xrightarrow{n \to +\infty} \psi$ in C^k, $k \gg 1$, solving (1)

Note that the higher order ~~...~~ energy estimates for the sequence of wave equations

$$\|\psi_n\|_{L^\infty_t H^k(\Sigma_t)} \leq C \, \|F_n\|_{H^{k-1}(R)}$$

uniform in n!!

So: the sequence ψ_n is uniformly bounded in $L^\infty_t H^k_x$

Rellich–Kondrachov theorem: $H^{m_1} \hookrightarrow H^{m_2}$ is compact when $m_1 < m_2$ (on a compact domain)

So ψ_n is a pre-compact sequence in $L^\infty_t H^{k-1}_x$

$\rightarrow \psi_n \xrightarrow{L^\infty_t H^{k-1}_x} \psi$ along a subsequence.

By the Sobolev embedding theorem (which we will prove shortly):

$$\|f\|_{C^k} \lesssim \|f\|_{H^m} \text{ if } m > k + \lceil \tfrac{n}{2} \rceil$$

So $\psi_n \xrightarrow{L^\infty_t C^2_x} \psi$

So ψ solves $\begin{cases} \Box_g \psi = F \\ \psi|_{\Sigma_0} = 0, \ \hat{n}(\psi)|_{\Sigma_1} = 0 \end{cases}$ $\square$

<u>Corollary</u>:

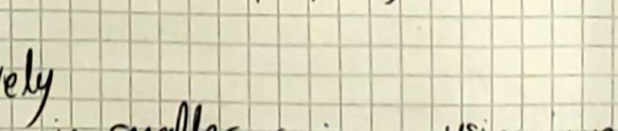
Let (M, g) be globally hyperbolic with Cauchy hypersurface Σ.

Then $\begin{cases} \Box_g \psi = F \\ \psi|_{\Sigma} = \psi_0, \ \hat{n}(\psi)|_{\Sigma} = \psi_1 \end{cases}$

is <u>globally</u> well-posed. Moreover, the domain of dependence property holds.

<u>Sketch of proof</u>: Inductively

By gluing the solution in smaller regions, using uniqueness for the overlaps:



The Sobolev estimate:

Def: Let $f : \mathbb{R}^n \to \mathbb{R}$

$$\| f \|_{H^k} = \left(\sum_{|a|=0}^{k} \int_{\mathbb{R}^n} |\partial^a f|^2 dx \right)^{1/2}$$

$$\partial^a f = \partial_1^{a_1} \cdots \partial_n^{a_n} f, \quad |a| = a_1 + \dots + a_n$$

and $$\| f \|_{\dot{H}^k} = \left(\sum_{|a|=k} \int_{\mathbb{R}^n} |\partial^a f|^2 dx \right)^{1/2}$$

Homogeneous Sobolev (semi)-norm.

Sobolev estimate: If f is smooth and comp. supported:

$$\| f \|_{L^\infty} \leqslant C \, \| f \|_{H^k} \quad \text{if} \quad k > \frac{n}{2}$$

Independent of f

Rmk: So it holds for any function for which $\| \cdot \|_{H^k}$ is bounded, due to the density of smooth & comp. supported functions.

Proof: Fourier transform:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\xi} f(x) dx$$

Inverse: $$f(x) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\xi} \hat{f}(\xi) d\xi$$

So: $$\| f \|_{L^\infty} \leq \| \hat{f} \|_{L^1}$$

And: $$\| f \|_{L^2} = \| \hat{f} \|_{L^2}, \quad \widehat{\partial^a f} = (i\xi)^a \hat{f}$$

So $\|f\|_{H^k} \sim \|(1+|\xi|)^k \hat{f}\|_{L^2}$

Hence: $\|f\|_{L^\infty} \lesssim \|\hat{f}\|_{L^1} = \int_{\mathbb{R}^n} |\hat{f}|\, d\xi = \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^k} (1+|\xi|)^k |\hat{f}|\, d\xi$

$$\leq \underbrace{\left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{2k}} d\xi\right)^{1/2}}_{< +\infty \text{ if } 2k > n} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2k} |\hat{f}|^2\, d\xi\right)^{1/2}$$

$\square$

<u>Non-linear wave equations</u>:

General quasilinear wave equation

$$\Box_{g(\psi,\partial\psi)} \psi = N(\psi, \partial\psi)$$

In local coordinates: $g_{ij}(\psi, \partial\psi)$ is Lorentzian.

We want to study the initial value problem for initial data posed on a hypersurface Σ which is spacelike with respect to $g(\psi, \partial\psi)|_\Sigma$

determined by the initial data, i.e. $\psi|_\Sigma, \partial\psi|_\Sigma$

We will consider the model example on $\mathbb{R}^{n+1}$

$$\partial_\alpha \left(G^{\alpha\beta}(\psi)\, \partial_\beta \psi\right) = N(\psi, \partial\psi)$$

where $G^{\alpha\beta}(0) = \eta^{\alpha\beta}$, $|G^{\alpha\beta} - \eta^{\alpha\beta}| < \frac{1}{10} \cdot \frac{1}{n}$

Note: $t=0$ is spacelike, since $G^{00}<0$

here $G^{\alpha\beta}$ is the inverse of a metric.

(Recall that, for a general Lor. metric

$$\Box_g f = \frac{1}{\sqrt{-\det g}}\partial_\alpha\left(\sqrt{-\det g}\, g^{\alpha\beta}\partial_\beta f\right),\ )$$

<u>Local well-posedness theorem for quasilinear wave equations:</u>

<u>Thm</u>: Consider the initial value problem on $\mathbb{R}^{n+1}$:

$$\begin{cases} \partial_\alpha\left(G^{\alpha\beta}(\psi)\,\partial_\beta\psi\right) = N(\psi,\partial\psi) \\ \psi|_{t=0} = \psi_0,\quad \partial_t\psi|_{t=0} = \psi_1 \end{cases},$$

where $|G^{\alpha\beta}-\eta^{\alpha\beta}| < \frac{1}{10}\cdot\frac{1}{n}$. Let also $k>2n+2$.

If $M = \|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$, then $\exists\, T=T(M)>0$ and a <u>unique solution</u> ψ on $[0,T]\times\mathbb{R}^n$, such that

$$\|\psi\|^2_{L^\infty_t H^k} + \|\partial_t\psi\|^2_{L^\infty_t H^{k-1}} \leq C M.$$

← Independent of M,ψ

Moreover, ψ depends continuously on (ψ_0,ψ_1).

<u>Proof</u>: <u>Existence</u>:

We will use a Picard iteration scheme: